

Beyond Integer Dimensions:

A Motivation Towards Fractal Geometry

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Abstract

What does it mean for a set to have a dimension? Smooth curves are 1-dimensional, smooth surfaces are 2-dimensional and solids are 3-dimensional. But in the nineteenth and early twentieth century mathematicians discovered objects for which this intuition is no longer enough. We get to meet some of these "mathematical monsters", including the Weierstrass function, the Cantor set and the Koch snowflake, and see why ordinary notions of length and size fail for them. Using the idea of scaling, we then arrive at Hausdorff measure and Hausdorff dimension, and briefly see how these ideas led to Fractal geometry and how it connects with Geometric measure theory.

1 Mathematical Monsters

For simple geometric objects, our intuition about dimension works very well.

A line has dimension 1, a square has dimension 2, and a solid cube has dimension 3. Also their usual notions of length, area and volume also as we expect.

But in the nineteenth century, mathematicians began to construct examples which seemed to contradict our geometric intuition.

The first example which contradicted our geometric intuition was the Weierstrass function.

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^{-k/2} \sin\left(\left(\frac{3}{2}\right)^k t\right).$$

The graph of this function looks like

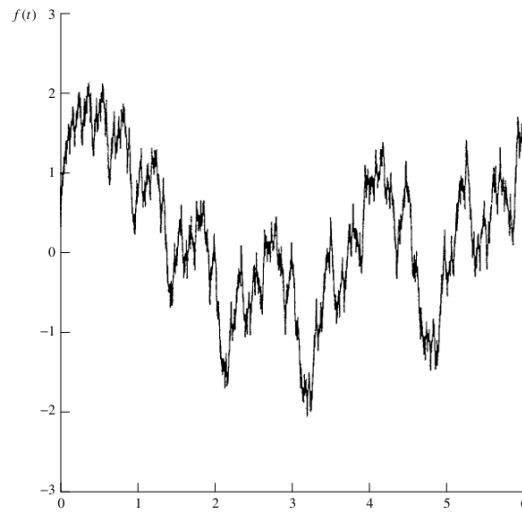


Figure 1: The Weierstrass function $f(t) = \sum_{k=0}^{\infty} (3/2)^{-k} \sin((3/2)^k t)$. It is continuous everywhere but differentiable nowhere.

It is continuous everywhere but differentiable nowhere. This was quite surprising at the time. We tend to think of a continuous curve as something that, if we zoom in enough, should look smooth and eventually have a tangent. The Weierstrass function showed that this intuition can completely fail: no matter how much we zoom in, there is no tangent anywhere

To many mathematicians of the time, these examples seemed like “monsters”. Yet they could not be ignored: they forced a more careful look at what we actually mean by size, dimension and geometry.

2 The Cantor Set

One of the most famous examples is the Cantor set.

Start with the interval $[0, 1]$. Remove the middle third. From each of the two remaining intervals, remove its middle third, and continue this process indefinitely.

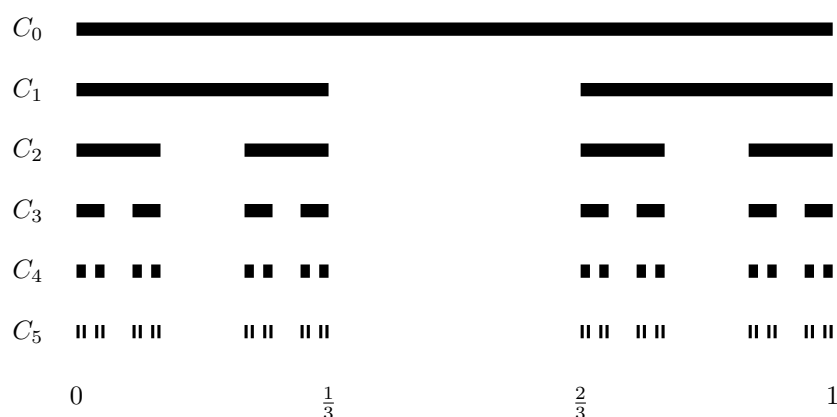


Figure 2: The first five iterations $C_0, \dots, C_5$ in the construction of the Cantor set. At each step the open middle third of every remaining interval is removed.

The resulting set is the Cantor set, which we denote by C . Some of its properties are:

- C is compact, i.e. closed and bounded.
- C is uncountable.
- C contains no interval, i.e. it has an empty interior.
- C has Lebesgue measure zero.

The last two properties are particularly interesting.

The Cantor set contains infinitely many points, in fact uncountably many, but its total length is zero or more precisely it has measure 0.

So if we try to understand the “size” of the Cantor set only using ordinary length, we lose a lot of information.

This leads to a natural question:

What is the dimension of the Cantor set?

It is clearly not behaving like an interval, but is it really zero-dimensional?

3 What Should Dimension Mean?

Before trying to define a new notion of dimension, let us look at how ordinary objects behave under scaling.

Suppose we cover an object by small pieces of size roughly r . Let $N(r)$ be the number of such objects.

For a line segment of unit length, we need about

$$N(r) \sim r^{-1}$$

pieces, precisely intervals of length $0 < r < 1$

For a unit square, we need about

$$N(r) \sim r^{-2}$$

pieces, i.e. squares of side $0 < r < 1$

Similarly, for a unit cube,

$$N(r) \sim r^{-3}.$$

Now, this suggests that dimension is related to the exponent describing how the number of pieces changes as the scale becomes smaller. It can also be understood as the rate at which the complexity of an object grows when we zoom in.

Heuristically,

Dimension is an exponent describing how the number of pieces needed to cover an object changes with scale.

Thus if the dimension is d then we would expect the scaling condition to be

$$N(r) \sim cr^{-d}$$

where c and d are constants. Now let us apply this idea to the Cantor set.

After n steps of the construction, there are 2^n intervals, each of length 3^{-n} .

If the dimension is s , we would expect

$$2^n \approx (3^{-n})^{-s}.$$

Therefore,

$$2^n = 3^{ns},$$

and hence

$$s = \frac{\log 2}{\log 3}.$$

This number is approximately 0.6309, So the Cantor set suggests that there should be a meaningful notion of dimension which does not have to be an integer.

But we have only used a scaling argument. Can we turn this idea into a proper definition?

4 The Koch Snowflake

Another important example is the Koch curve.

Start with a line segment. Divide it into three equal parts, remove the middle part, and replace it by two segments forming an equilateral triangle. Repeat this construction on every segment.

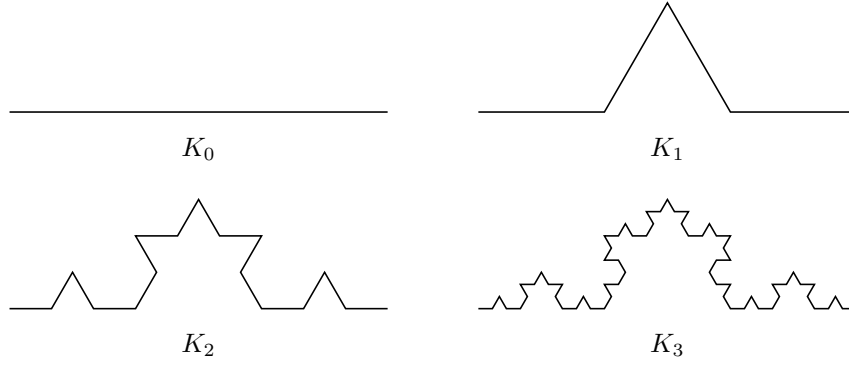


Figure 3: The first four iterations $K_0, \dots, K_3$ in the construction of the Koch curve. At each step the middle third of every segment is replaced by two sides of an equilateral triangle.

At every stage the curve becomes more complicated.
After n steps,

$$\text{number of segments} = 4^n,$$

while the length of each segment is

$$3^{-n}.$$

The total length after n steps is therefore

$$4^n 3^{-n} = \left(\frac{4}{3}\right)^n,$$

which tends to infinity.

Thus the von Koch curve has infinite length, even though it is contained in a bounded region. If we apply the same construction to the three sides of an equilateral triangle, we obtain the *Koch snowflake*. Its boundary has infinite length, yet the region it encloses has finite, non-zero area. This area is found by adding the areas of all the triangles added at each step to the area of the original triangle.

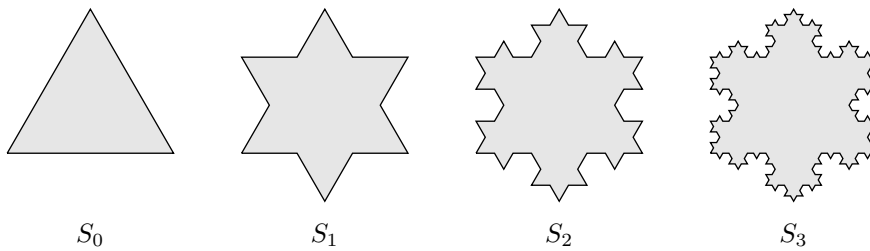


Figure 4: The first four iterations $S_0, \dots, S_3$ of the Koch snowflake. Starting from an equilateral triangle, the construction of the Koch curve is applied to each of its three sides.

Again, ordinary length does not seem to capture the geometry very well. Also geometrically its quite contradicting that a curve of infinite length covers a finite area. But here we have an example for it.

Now the scaling behavior from earlier gives us a possible dimension.

If the dimension is s , then for the von Koch curve

$$4^n \approx (3^{-n})^{-s}.$$

Hence

$$s = \frac{\log 4}{\log 3} \approx 1.2619.$$

This is bigger than 1 but smaller than 2.

So the von Koch curve behaves, in some sense, like something between a curve and a two-dimensional object.

5 Hausdorff Measure

The scaling calculations suggest that we should measure a set using arbitrarily small pieces.

For $s \geq 0$, consider coverings of a set E by sets U_i whose diameters are at most δ . Define

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_i (\text{diam } U_i)^s : E \subseteq \bigcup_i U_i, 0 < \text{diam } U_i \leq \delta \right\}.$$

We then let δ tend to zero and define

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E).$$

This is the s -dimensional Hausdorff outer measure.

The important point is that s does not have to be an integer.

For $s = 0$, this is just the counting measure or the cardinality.

For $s = 1$, this is related to length.

For $s = 2$, it is related to area.

In fact at non-negative integers it's just related to the Lebesgue measure. But we can also consider values such as

$$s = 0.6309 \quad \text{or} \quad s = 1.2619.$$

This is exactly what is needed for sets such as the Cantor set and the Koch curve.

6 Hausdorff Dimension

Hausdorff measure allows us to define dimension.

For a set E , there is a critical value s at which the behavior of $\mathcal{H}^s(E)$ changes. We will have that

$$\mathcal{H}^t(E) = \infty \quad 0 \leq t < s,$$

and

$$\mathcal{H}^s(E) = 0 \quad t > s.$$

The transition point is called the Hausdorff dimension:

$$\dim_H E = \inf\{s : \mathcal{H}^s(E) = 0\}.$$

Equivalently,

$$\dim_H E = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

Thus, Hausdorff dimension provides a way to assign dimensions to sets that do not fit into the usual integer-dimensional picture. Moreover, it retains several natural properties we expect a notion of dimension to have, such as monotonicity and invariance under suitable transformations, while allowing non-integer values.

7 The Cantor Set Revisited

Let us return to the Cantor set.

At the n -th stage, we have 2^n intervals of length 3^{-n} .

For an exponent s , the corresponding quantity is

$$2^n(3^{-n})^s = (2 \cdot 3^{-s})^n.$$

The critical value occurs when

$$2 \cdot 3^{-s} = 1.$$

Therefore,

$$s = \frac{\log 2}{\log 3}.$$

Hence

$$\dim_H C = \frac{\log 2}{\log 3}.$$

So although the Cantor set has Lebesgue measure zero, its Hausdorff dimension is strictly between 0 and 1.

8 A New Notion of Dimension

The examples of the Cantor set and the Koch curve show why Hausdorff dimension is useful.

The Cantor set has

$$\dim_H C = \frac{\log 2}{\log 3} < 1,$$

while the von Koch curve has

$$\dim_H K = \frac{\log 4}{\log 3} > 1.$$

This gives us a much more flexible notion of dimension.

It is also important to note that a set having a non-negative integer Hausdorff dimension doesn't simply imply that the Lebesgue measure has to be positive and finite.

For example, a subset of $\mathbb{R}$ can have Lebesgue measure zero but still have Hausdorff dimension 1. Construction of thin Cantor like sets can give us many examples. By thin what I mean to say the length of the approximating sets tends to zero, but it does so slowly enough that the limiting set retains full Hausdorff dimension.

Thus "measure zero" does not necessarily mean "small" from the point of view of Hausdorff dimension.

9 From Mathematical Monsters to Fractals

These examples were developed long before the modern word "fractal" became common.

In 1975, Benoit Mandelbrot introduced the term "fractal" and helped bring together many of these ideas and these so called "monsters" under the subject of fractal geometry. The word fractal came from the Latin adjective "fractus" and the corresponding Latin verb is "frangere" which means "to break" to create irregular fragments.

Self-similar constructions such as the Cantor set and Koch curve are some of the simplest examples.

But fractal behavior occurs far beyond these examples, including in dynamical systems, probability, harmonic analysis, and in many natural phenomena such as Brownian motion. The basic idea is that complicated geometric objects can have structure at many different scales.

10 Connection with Geometric Measure Theory

Hausdorff measure and Hausdorff dimension are also fundamental tools in geometric measure theory.

GMT studies geometric properties of sets and measures, especially when the sets are too irregular for classical geometric methods.

Questions such as

- How large is a complicated set?
- What is its dimension?
- How does a measure behave on different scales?
- How can we describe irregular or fractal sets?

are central to the subject.

Thus what began as a collection of strange and pathological examples eventually led to a new way of thinking about geometry.

11 Conclusion

The story can be summarized in a simple chain:

Mathematical Monsters $\longrightarrow$ New Notions of Size $\longrightarrow$ Hausdorff Dimension $\longrightarrow$ Fractal Geometry

The main lesson is that dimension does not have to be an integer.

Once we start looking at how objects behave at smaller and smaller scales, a much richer geometry appears.

References

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